Stat 155 Lecture 17 Notes

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1 Pigou Networks and Cooperative Games

1.1 Pigou networks

Last time, we studied the price of anarchy for linear and affine latencies. More generally, suppose we allow latency functions from some class $\mathcal{L}$. So far, we have considered the following classes:

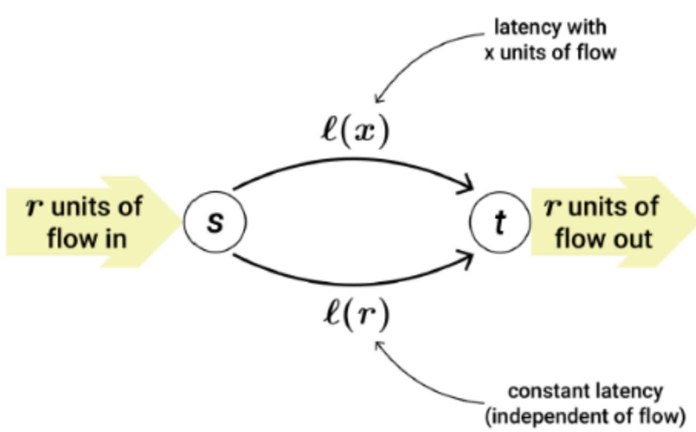
$$\mathcal{L}_{\text{linear}} = \{x \mapsto ax : a \geq 0\}$$

$$\mathcal{L}_{\text{affine}} = \{x \mapsto ax + b : a, b \geq 0\}.$$

What about the class

$$\mathcal{L} = \{x \mapsto \sum_d a_d x^d : a_d \geq 0\}$$

of polynomial latencies? We will insist that latency functions are non-negative an non-decreasing. It turns out that the price of anarchy in an arbitrary network with latency functions chosen from $\mathcal{L}$ is at most the price of anarchy in a certain small network with these latency functions: a Pigou network.



Definition 1.1. The *Pigou price of anarchy* is the price of anarchy for this network with latency function and total flow r:

$$\alpha_r(\ell) = \frac{r\ell(r)}{\min_{0\le x\le r} x\ell(x) + (r-x)\ell(r)}.$$

Theorem 1.1. *For any network with latency functions from $\mathcal{L}$ and total flow 1, the price of anarchy is no more than*

$$\mathcal{A}_r(\mathcal{L}) := \max_{0\le r\le 1} \max_{\ell\in\mathcal{L}} \alpha_r(\ell).$$

Proof.

$$\begin{aligned}
L(f) &= \sum_e F_e\ell_e(F_e) \\
&= \sum_e \left[\frac{F_e\ell_e(F_e)}{\min_{0\le x\le r}(x\ell_e(x) + (F_e-x)\ell_e(F_e))}\right] \min_{0\le x\le r}(x\ell_e(x) + (F_e-x)\ell_e(F_e)) \\
&= \sum_e \alpha_{F_e}(\ell_e) \min_{0\le x\le r}(x\ell_e(x) + (F_e-x)\ell_e(F_e)) \\
&\le \sum_e \alpha_r(\ell_e)(F_e^*\ell_e(F_e^*) + (F_e-F_e^*)\ell_e(F_e)) \\
&\le \max_{r\in[0,1],\ell\in\mathcal{L}} \alpha_{F_e}(\ell_e)\left(\sum_e F_e^*\ell_e(F_e^*) + \sum_e (F_e-F_e^*)\ell_e(F_e)\right) \\
&\le \max_{r\in[0,1],\ell\in\mathcal{L}} \alpha_r(\ell_e) \sum_e F_e^*\ell_e(F_e^*) \\
&= \max_{r\in[0,1],\ell\in\mathcal{L}} \alpha_r(\ell_e) L(f^*).
\end{aligned}$$

□

Example 1.1. Consider a Pigou network with $r=1$, nonlinear latency $\ell_e(x) = x^d$, and $\ell(r) = 1$. The Nash equilibrium flow is concentrated completely on the top edge: $L(f) = 1$. The socially optimal flow gives:

$$L(f^*) = \min_x(1 - x + x^{d+!}) = 1 - d(d+1)^{(d+1)/d}.$$

The price of anarchy is

$$\frac{1}{1-d(d+1)^{(d+1)/d}} \sim \frac{d}{\ln(d)}.$$

What about $\alpha_r(\ell_e)$? Let

$$g(x) = x\ell(x) + (r-x)\ell(r).$$

Taking the derivative to zero, we get $x^* = r/(d+1)^{1/d}$ is the point where g attains the minimum. So

$$\alpha_r(\ell_e) = \frac{r\ell(r)}{g(x^*)} = \frac{r^{d+1}}{\frac{r^{d+1}}{(d+1)^{(d+1)/d}} - r^{d+1} + \frac{r^{d+1}}{(d+1)^{1/d}}} \sim \frac{d}{\log d}.$$

1.2 Cooperative games

Let's review noncooperative games. Players play their strategies simultaneously. They might communicate (or see a common signal, e.g. a traffic signal), but there is no enforced agreement. The natural solution concepts are Nash equilibrium and correlated equilibrium. What if the players can cooperate?

In cooperative games, players can make binding agreements. For example, in the prisoner's dilemma, the prisoners can make an agreement not to confess. Both players gain from an enforceable agreement not to confess. There are two types of agreements.

Definition 1.2. An agreement has *transferable utility* if the players agree what strategies to play and what additional side payments are to be made.

Definition 1.3. An agreement has *nontransferable utility* if the players choose a joint strategy, but there are no side payments.

Example 1.2. Consider the game with payoff bimatrix

$$\begin{pmatrix} (2,2) & (6,2) & (1,2) \\ (4,3) & (3,6) & (5,5) \end{pmatrix}.$$

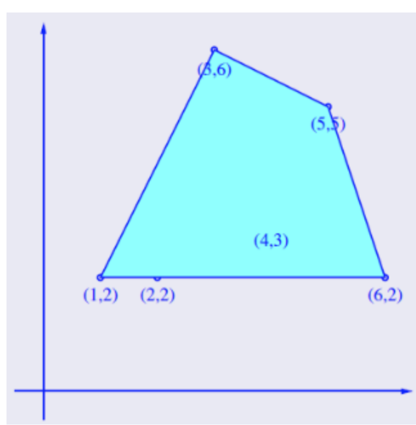
What should the players agree to play if they cannot transfer utility? Try it with a friend![1]

Definition 1.4. The set of payoff vectors that the two players can achieve is called the *feasible set*.

With nontransferable utility, the feasible set is the convex hull of the entries in the payoff bimatrix.

Definition 1.5. A feasible payoff vector (v_1, v_2) is *Pareto optimal* if the only feasible payoff vector (v_1', v_2') with $v_1' \geq v_1$ and $v_2' \geq v_2$ is $(v_1', v_2') = (v_1, v_2)$.

Example 1.3. In our cooperative game example, the feasible region is



[1]If you do not have any friends, send me an email, and I will play this game with you.

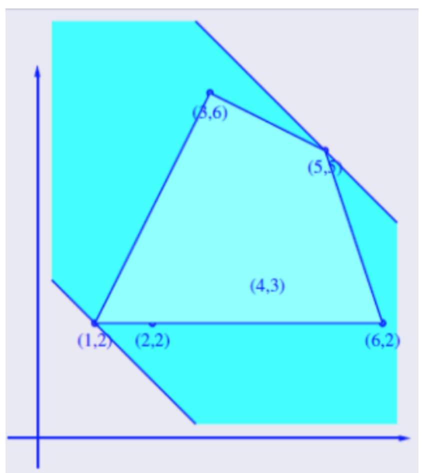
The Pareto boundary is the part of the feasible region with nothing to the right of or above it.

Example 1.4. Consider the same payoff bimatrix as before, but now assume that the payoff is in dollars.

$$\begin{pmatrix} (2,2) & (6,2) & (1,2) \\ (4,3) & (3,6) & (5,5) \end{pmatrix}.$$

The two players need to agree on what they will play, and they can pay each other to incentivize certain strategies. What is the best total payoff that can be shared? How should it be shared? Try it with a friend!

With transferable utility, the players can choose to shift a payoff vector. For example, suppose a pure strategy pair gives payoff $(a_{i,j}, b_{i,j})$. Suppose the players agree to play it, and Player 1 will give Player 2 a payment of p. The payment shifts the payoff vector from $(a_{i,j}, b_{i,j})$ to $(a_{i,j} - p, b_{i,j} - p)$. The feasible region looks like this:



Here, the Pareto boundary is the line $y = -x + 10$.